



# Optimization: Squares, Constraints, and Symmetry

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## A Quadratic with a Floor

Find the smallest possible value of

$$x^2 + 2xy + 3y^2 - 4x - 8y + 10$$

for real  $x, y$ . At which points is it attained?

## A Plane and a Sphere

- (a) Find the largest and smallest values of  $x + 2y + 2z$  on  $x^2 + y^2 + z^2 = 1$ .
- (b) Among points satisfying  $x + 2y + 2z = 6$ , find the point closest to the origin.

## The Range of a Moving Coordinate

- (a) Find the range of  $5 \cos t + 12 \sin t$  as  $t$  runs through the real numbers.
- (b) For arbitrary real  $a, b, c$ , find the largest and smallest values of  $c + a \cos t + b \sin t$ .

## A Quadratic on a Circle

Find the largest and smallest values of  $x^2 + 2xy + 3y^2$  subject to  $x^2 + y^2 = 1$ . Describe the points where each value is attained.

## A Circle with Many Coordinates

For vectors in  $\mathbb{R}^n$ , write  $u \cdot v = \sum u_i v_i$  and  $\|u\|^2 = u \cdot u$ . Suppose  $\|u\| = \|v\| = 1$  and  $u \cdot v = 0$ . Let  $c \in \mathbb{R}^n$ ,  $R > 0$ , and

$$P(t) = c + R(u \cos t + v \sin t).$$

- (a) Show that  $P(t)$  traces a circle of radius  $R$  in a plane through  $c$ .
- (b) Find the range of its  $i$ -th coordinate. What is the sum, over all coordinates, of the squares of these range lengths?

### A Cube in a Cone

A right circular cone has base radius 1 and height 3. A cube is inscribed in it with one face contained in the base of the cone. What is the side length of the cube?

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*Source: Putnam 1998 A1*

### Packing Two Squares

Find the least number  $A$  such that any two squares of combined area 1 can be packed, without interior overlap, into a rectangle of area  $A$ . The squares' sides must be parallel to the rectangle's sides.

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*Source: Putnam 1996 A1*

### Three Coordinates, One Product

For nonnegative  $x, y, z$  satisfying  $x^2 + y^2 + z^2 = 3$ , find the largest possible value of  $xyz$  and all equality cases.

### A Circle in a Four-Dimensional Cube

What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1? Here the hypercube may be taken to be  $[0, 1]^4$ ; a circle is an ordinary Euclidean circle lying in a two-dimensional affine plane in  $\mathbb{R}^4$ .

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*Source: Putnam 2008 B3*